What can be more challenging than square-rooting from squared things?

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Roots is one of these mathematical topics that students encounter multiple times in their studies of mathematics. Growing from the commognitive framework, this study is concerned with intra-commognitive conflicts where the same discursant develops incommensurable discourses on 'the same' mathematical object (square roots, in this case) and endorses seemingly conflicting narratives about it. These conflicts were explored among eleven pre-academic students who worked individually on a multiple-question assignment asking them to extract square roots from squared numbers and expressions. Three intra-commognitive conflicts are presented in this paper: a conflict of perfect squares and squared inputs, a conflict of stand-alone and incorporated roots, and a conflict of expressions and equations.

Roots is one of these mathematical topics that accompany students' studies all the way through, from secondary school to university. Typically introduced in the context of integers, roots are gradually extended to fractions, parametric expressions, equations, and functions. Through the Pythagoras theorem and the law of cosines, roots find their way into geometry and trigonometry lessons. In the university setting, roots are often used to exemplify more advanced topics, such as inverse functions in real analysis and multi-valued functions in complex analysis. Thus, it is barely surprising that school curricula and pre-academic programmes in many countries dedicate dozens of teaching hours to develop students' proficiency with this fundamental mathematical topic. Such a dedicated didactic investment opens a space for empirical research on students' grasp of roots.

The study reported in this paper is a part of a larger project on teaching and learning of roots in school and university classrooms (see Kontorovich, 2018a and b for a previous report to MERGA community). The project has been initiated to enrich the palette of empirically identified challenges that students encounter with roots in different mathematics areas. Secondary algebra is under scrutiny in this study.

The focus of this study has been instigated by my personal experience with undergraduates, high-school students, and teachers, whom I often ask to simplify $\sqrt{x^2}$. Despite profound mathematical knowledge of many of them, x and $\pm x$ are the most popular answers, when $|x|$ is rarely mentioned. My experience comes from Israel and New Zealand, but it echoes with the observations of Roach, Gibson, and Weber (2004) in their algebra, pre-calculus, and calculus classrooms in the US: as a response to $\sqrt{25}$, most of their freshmen responded with ± 5. Crisan (2014) reports on similar tendencies in a group of pre-service secondary mathematics teachers in the UK, where the question "$\sqrt{25y^2}$=?" divided the group between those who insisted on "$\sqrt{y^2} = y$", and those who believed in "$\sqrt{y^2} = \pm y$".

The generation of similar answers to arguably basic questions by people with substantially different mathematical backgrounds constitutes one motivation for this study. Another motivation is concerned with conflicts that the above unconventional answers might be expected to evoke among square-rooters. Indeed, if one assumes that the square root of a squared radicand is the initial input itself, then what happens when this input is negative (e.g., $\sqrt{(-6)^2}$)? How does this assumption align with a commonly emphasized feature of even roots producing non-negative outcomes? If another square-rooter is convinced that the

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concept produces two opposite values, how would she justify $\sqrt{x^2} - \sqrt{x^2}$ being equal to 0 and not 0 and $\pm 2x$? This study explores conflicts of this ilk as they emerge from the analysis of the written assignments of pre-academic students who square-rooted from squared numbers and parametric expressions.

Theoretical Framing

This study is nested in the commognitive framework of learning (Sfard, 2008), which has become a widely accepted discursive approach in mathematics education, especially at the postsecondary level (e.g., Nardi, Ryve, Stadler, & Viirman, 2014). The framework is concerned with human discourses, which are defined as "different types of communication, set apart by their objects, the kinds of mediators used, and the rules followed by participants and thus defining different communities of communicating actors" (Sfard, 2008, p. 93). From this standpoint, mathematics is conceptualized as a discourse with four specific characteristics: *words* (e.g., "square roots") and their use; *visual mediators* (e.g., "$\sqrt{\ }$") and their use; generally endorsed *narratives* (e.g., "$\sqrt{x^2} = |x|$"); and recurrent *routines* (e.g., extracting roots).

The framework's constructs of discursive objects, routines, and conflicts are central to this study. The first two of them are briefly presented next. The third construct is explained and further developed.

Mathematical objects

The illusiveness and intangibility of mathematical objects have been acknowledged by many schools of thought. Commognition posits that when communicating, we operate with perceptually accessible *signifiers* (e.g., words, symbols, gestures) that are realized into other signifiers, which are realized further forming *realization trees*. In this way, "[t]he discursive object signified by S in a given discourse on S is the realization tree of S within this discourse" (ibid, p. 166). This definition turns mathematical objects into discursive, personalized and contextualized constructs. Let us consider an example: a student was assigned with a question "What is the square root of x^2?" and she wrote "$\sqrt{x^2}$". With the commognitive lens, this occurrence suggests that that the student's realization tree of square roots contains at least three signifiers that she treated as equivalent in this context: the written "square root", phonetical "skweə ruːt", and the radix symbol.

Discursive routines

The notion of routine reflects one of the central premises of commognition, asserting that discourses are patterned and rule-driven activities that allow discursants to be efficient in situations that they consider as similar. Lavie, Steiner, and Sfard (online) contextualize routines in a *task situation* that pertains to any setting, where a discursant considers herself bound to do something. Her capability to act in a new situation is explained by harking back to *precedents* – task situations that she views as sufficiently similar to the present one. Such a view justifies recreating patterns that are familiar from the discursant's experience or from the experiences of others. The choice of specific precedents occurs with the help of *precedent identifiers*, which are features of a current task situation that a discursant considers sufficient to make the link to past precedents.

Lavie et al. (online) propose that "a *task*, as understood by a person in a given task situation, is the set of all the characteristics of the precedent events that she considers as requiring replication" (my italics, p. 9). Due to the unavoidable difference between precedents and a current task situation, the replication will preserve some of the past actions and change some others. This premise is reflected in a *procedure*, which is “the prescription for action that fits both the present performance and those on which it was modeled” (ibid, p. 9). Eventually, the researchers posit that a “*routine* performed in a given task situation by a given person is the task, as seen by the performer, together with the procedure she executed to perform the task” (my italics, p. 9).

Intra-commognitive conflicts

Sfard (2008) defines a commognitive conflict as “the encounter between interlocutors who use the same mathematical signifiers (words or written symbols) in different ways or perform the same mathematical tasks according to differing rules” (p. 161). The presented divide in Crisan (2014) features such a conflict since the teachers’ realization routines for “$\sqrt{25y^2}$=?” differed in their procedures and resulted in conflicting outcomes. These routines characterize different discourses that do not share a well-defined set of rules that would allow the teachers from both “square-rooting camps” to resolve the conflict deductively. Sfard terms such discourses as *incommensurable* and explains that “two narratives that originate in incommensurable discourses cannot automatically count as mutually exclusive even if they sound contradictory” (p. 258).

Commognitive conflicts are often sought (and found) in situations where different interlocutors are involved. Yet, Sfard attributes commognitive conflicts to discourses, which may or may not be ascribed to different people. Then, I propose the notion of *intra-commognitive conflict* for referring to situations where the same discursant endorses conflicting narratives or where there is a potential for generating such narratives based on the observable rules of her mathematical discourses. To be explicit, an intra-commognitive conflict is an analytical construct, when the discursant herself may not be aware of the conflict that an analyst discerns, nor does she necessarily experience it as such. In this way, while “the differences in metarules […] are the source of the [commognitive] conflict find their explicit, most salient expression in the fact that different participants endorse contradicting narratives” (Sfard, 2008, p. 256), intra-commognitive conflicts might be not easy for a discursant to spot on her own.

The study

Research aim and question

This study aims *to explore intra-commognitive conflicts that underlie students’ discourses on square roots*. Sfard (2008) notes that each communicational medium – verbal, gestural, iconic, symbolic – “has its own discourse that supports its unique set of narratives” (p. 156), making some discursive moves “easier to perform than some others, depending on the type of “materials” in which they are implemented” (p. 156). The written medium is in the focus of this study.

Vygotsky (1986) elaborates on the distinctive nature of the written speech and on its contribution to the development of scientific concepts. While an oral speech provides room for intonations, vocal emphases, gestures, and other hints to support interlocutors’ communication, written speech can be viewed a monologue directed towards someone at the time of their physical absence. In Vygotsky’s (1986) words,

> “Written speech is deployed to its fullest extent, more complete than oral speech. Inner speech is almost entirely predicative because the situation, the thinker always knows the subject of thought. Written speech, on the contrary, must explain the situation fully in order to be intelligible. The change from maximally compact inner speech to maximally detailed written speech requires what might be called deliberate semantics – deliberate structuring of the web of meaning” (p. 182).

To sum, the written medium summons speech (or a discourse, in commognitive terms) that is formal and precise. Both features are essential for mature students, whose writing constitutes a dominant communicational channel, through which their mathematical discourses are developed, captured, and assessed. Accordingly, the research question instigating this study is "*What intra-commognitive conflicts can be detected from students' written square-rooting?*"

Participants, data, and analysis

The data for this study came from 11 eighteen- and nineteen-year-olds, who enrolled in a pre-academic program in a large technological university in Israel. These students finished school with the minimal mathematical requirements of the national educational system, and they enrolled in the program to improve their qualifications and achievements in school subjects to get accepted to the universities and faculties of their choice. The particular pre-academic program reengages its students with high-school mathematics and with additional topics, such as arithmetic techniques. According to the Israeli school curriculum, roots are addressed in algebra, analytical geometry, pre-calculus functions, differentiation, integration, and vectors. At the time of data collection, the participating students re-covered the first topic from this list as part of their program's studies.

Research has noted that canonical discourses on roots do not always align with each other (e.g., Crisan, 2012; Roach et al., 2004). Thus, let me sketch the discourse that may be considered as canonical in the context of this study. Israeli curriculum defines b as a square root of a if $b^2 = a$, which ensures two square-roots for a positive number. The signifiers "the square root" and '$\sqrt{\ }$'-symbol are used for non-negative roots only. These definitions might ease on students' transitions between arithmetic, algebraic, and calculus discourses. For instance, the statement "$\sqrt{9} = 3$", is correct through the lens of algebra as well as when it is approached as a function $f(x) = \sqrt{x}$ at $x = 9$. Hence, while the participants had several mathematical discourses at their reach that could be leveraged for square-rooting, all of them could be expected to result in the same outcomes.

The data were collected with an assignment, fifteen questions of which requested the students to extract square roots from squared numbers and parametric expressions. The participants' mathematics teacher confirmed that the assignments' questions were not very different from the ones that were discussed in the classroom. The assignment was distributed in a regular mathematics lesson and the students worked on it individually without using calculators. While the work was not time-limited, all students submitted their assignments in less than 25 minutes.

The data analysis was shaped by the commognitive research principles (Sfard, 2008), which were employed through the constant comparison technique (Glaser & Strauss, 1967). Specifically, fine-grained comparisons were made within the responses of each student and in-between their assignments. The comparisons were targeted at indicating changes in students' use of symbols, words, and narratives, which served as a baseline for delineating students' discourses and potential conflicts between them.

Findings

Three intra-commognitive conflicts are reported herein: a conflict of perfect squares and squared inputs, a conflict of stand-alone and incorporated roots, and a conflict of expressions and equations. The first conflict is presented in more details to showcase the incommensurability of the discourses that gave rise to it. A similar incommensurability has been identified in the remaining conflicts, and then they are presented more briefly with a focus on their gist.

Conflict of perfect squares and squared inputs

Let us attend to some of the responses that Anna (pseudonym) submitted. In the questions with radicands presented as perfect squares, she preceded the radical symbol with the '±'-sign and responded with two opposite roots encapsulated under the '±' (see the left part of Figure 1 for example). In the questions with squared radicands, she started with converting the radical symbol to the power of half, followed by reducing the powers to 1, and concluded with the initially squared input (see the right part of Figure 1). More or less the same procedure was observed when Anna square-rooted from positive numbers and parametric expressions squared (e.g., "$\sqrt{10^2}$" and "$\sqrt{(-a)^2}$").

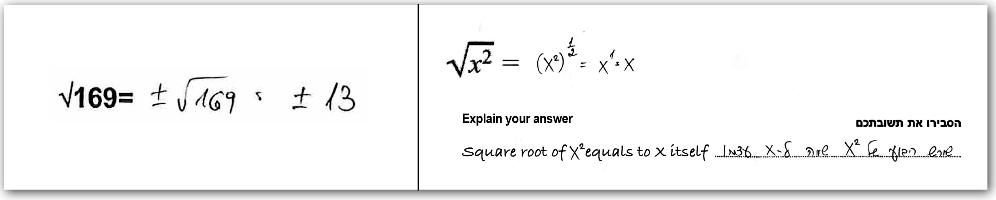

Figure 1. Illustration of a *conflict of perfect squares and squared inputs*.

The described patterns allow proposing that Anna developed (at least) two discourses on square roots differing in their characteristic symbols and routines. One discourse is distinguishable through natural numbers and Anna's usage of the '±' symbol. The square-rooting routine in this discourse occurs through a single realization step that entails two opposite outcomes. The other discourse brings integers and parameters under the same roof. There, square-rooting constitutes a more compound procedure, in which roots are realized into power notation eventually producing a single result. Anna alternated between these discourses as a response to the assigned task situation, and specifically, to the prompt under the radical symbol: perfect squares evoked the former discourse, while squared radicands gave rise to the latter one. Accordingly, I propose that the structure of the assigned prompts served as a precedent identifier signaling Anna in which discourse to engage.

Is it possible that capturing the compound realization chains in the latter discourse was a part of Anna's task, i.e. it reflects her understanding that the chain must be demonstrated but it was not necessary for her to obtain the final answers? Indeed, the assignment's questions asked students to explain their work, and what could be more explanatory than showing the path from the initial prompts to the end-products in full. While there are no univocal grounds to reject this possibility, two arguments can be offered for its unlikelihood. *First*, Anna's square-rooting and lack of explanations in a variety of task situations demonstrate that she was comfortable with providing "immediate" outcomes and ignoring explicit guidelines. Yet, this was observed in task situations with perfect squares and not with squared inputs. *Second*, the assignment contained numerous prompts with radicands in a squared form, and there Anna recreated a similar multi-step procedure. If these steps were unnecessary for her square-rooting, similarly to her classmates, she could have demonstrated it just a few times to signal her task setters that she is capable of substantiating her work.

In terms of discursive objects, the above considerations allow suggesting that in Anna's realization trees, perfect squares were linked to their square roots directly. Her not providing any verbal explanations in these task situations echoes with Sfard's (2008, p. 158) comment on realization procedures that might be difficult to explain, once they become embodied, automated, and evoked spontaneously upon encountering respective signifiers. In turn, Anna's branches of realization trees between squared radicands and their roots contained intermediate realizations that, perhaps, she could not avoid. Overall, Anna's assignment illustrates that one's discourses on square roots could differ not just in their characteristic routines but also in their degree of embodiment and automatization.

An intra-commognitive conflict can be discerned between Anna's discourses as they produce narratives that seem mutually exclusive. For instance, if "square root of x^2 is x itself", then what should the square root of $13^2 = 169$ be? Within the former discourse, Anna maintained that it is "± 13". The latter discourse, however, would advocate for 13 as the only answer. On a more general note, if one conceives parameters as encapsulations of numerical instances, then the parametric narratives of the latter discourse would be expected to act as object-level rules of the former one. Notably, Anna featured this logic in the right part of Figure 1, where she produced a narrative about parameters for substantiating her work with numbers. However, this logic held as long as the radicands were squared, and it did not extend further to perfect squares.

Conflict of stand-alone and incorporated roots

This conflict pertains to instances where the students seem to engage in incommensurable discourses depending on whether the task situation requested them to square-root only, or whether additional calculations were needed once the roots were extracted. Figure 2 juxtaposes two excerpts from the assignment of Betty who square-rooted differently from 169 and 81. While both radicands seem as different manifestations of the same mathematical object (e.g., a number, an integer, a perfect square), the procedures that Betty enacted entail seemingly conflicting outcomes. Indeed, if stand-alone roots are realized into two opposite results, how come that only one of them is acted on when roots become part of symbolic expressions with additional operations? Another manifestation of this conflict appeared when some students square-rooted differently from stand-alone parametric expressions and when these were incorporated in calculation exercises.

Conflict of expressions and equations

In the cases presented above, the students simplified the assigned prompts through square-rooting, which suggests that the tasks in which they engaged aligned with the ones intended by the questions. The conflict of expressions and equations emerged when the students stopped treating the assigned parametric prompts as algebraic expressions and embarked on them as equations. Figure 3 features such an instance with an excerpt from the assignment of Cindy. The left part of Figure 3 illustrates that as long as the questions contained operations with roots, she realized the radical symbols into simpler signifiers and utilized them further to simplify the assigned expressions. Once the prompts combined roots with addends that did not require simplification (like $2x$ in the right part of Figure 3), Cindy equalized the prompts to zero in attempt to determine the value of the parameter. This conflict can be positioned at the interface between students' discourses on expressions and equations, when characteristic routines in each discourse were targeted at outcomes that barely count as "the same" – simplified versions of the assigned expressions and numeric values of the parameter nullifying these expressions. Furthermore, the analysis of students' responses to a variety of task situations suggested that the structure of assigned prompts served as a precedent identifier for this conflict, and not the name of the parameter that a prompt involved. Indeed, the students who shifted between simplifying and equation-solving did so in disregard of whether the prompts contained letters from the beginning or the end of the alphabet.

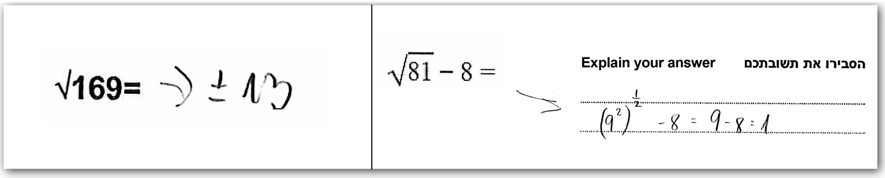

Figure 2. Illustration of a conflict of *stand-alone and incorporated roots*.

Discussion

With its focus on secondary algebra, this research report is part of a larger effort to enrich the palette of empirically indicated challenges with roots that school and university students encounter in different areas of mathematics (e.g., Kontorovich, 2018a, b). The theoretical contribution of this study pertains to the introduction of the construct of intra-commognitive conflicts. Mathematics education research in general, and commognition specifically, have been often concerned with discursive conflicts between "newcomers" to a mathematical discourse and its "oldtimers" (e.g., Crisan, 2014; Nardi et al., 2014; Roach et al., 2004; Sfard, 2008). Intra-commognitive conflicts draw attention to factual and potential conflicts between mathematical discourses of the same interlocutor.

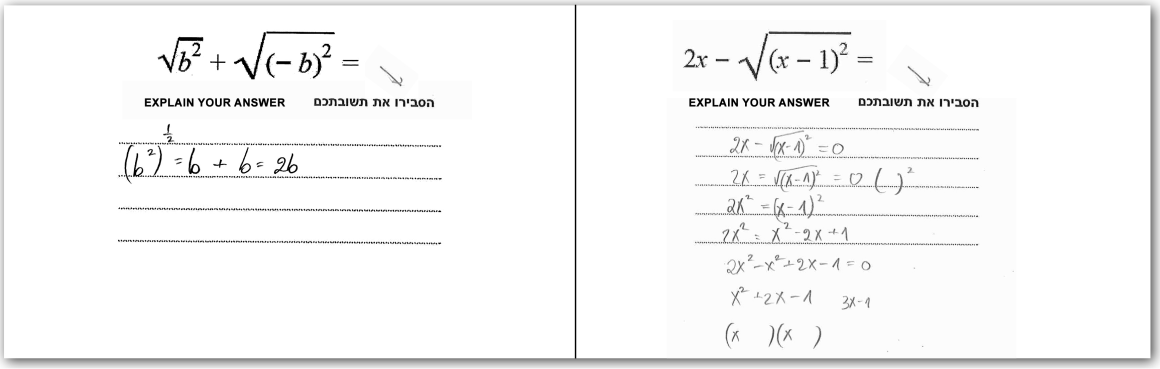

Figure 3. Illustration of a *conflict of expressions and equations*.

A similar attention to conflicts of this ilk is drawn in Alcock and Simpson (2011), who explored how undergraduates classify sequences of real numbers into increasing and decreasing. Coming from the concept image/concept definition underpinnings, their research showed that many students may lack what the researchers termed as "concept consistency" – a single mechanism for judging all assigned prompts. Similarly to students in this study who square-rooted differently from different roots, the participants in Alcock and Simpson (2011) alternated their classification approaches as a response to different sequences that they were given. Thus, the constructs of concept consistency and intra-commognitive conflicts can be viewed as analytical congeners growing from theories with epistemologically incommensurable foundations.

The conflicts identified in this study may be of practical interest to mathematics teachers, teacher educators, and textbook writers. Let me conclude with two comments in this regard. First, the conflicts emerged from postsecondary students, i.e. mathematics learners who intensively operated with roots for years in different mathematical areas and settings. These students join the undergraduates in Roach et al. (2004) and pre-service teachers in Crisan (2014) by demonstrating unresolved challenges with square-rooting from squared radicands. The exceptional resilience of these challenges to formal instruction evidence that learners may not overcome (or even notice) them without help from "oldtimers" of a canonical mathematical discourse. With this study, I hope to convince teachers and teacher educators in addressing these challenges explicitly in their teaching. Second, the nuanced distinctions that the participating students demonstrated between perfect squares and squared radicands, stand-alone and incorporated roots, and expressions and equations, emerged from a set of questions that some may label as "procedural" and "routine". When the voices against routine exercises sound louder than ever, this study points at their diagnostical potential of such exercises; a potential that teachers can leverage to promote mathematical discourses of their students.

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